

ON THE SECTIONAL CURVATURE ALONG RELATIVE EQUILIBRIA

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ABSTRACT. In this paper we explore a characterisation of relative equilibria in terms of sectional curvatures of the Jacobi-Maupertuis metric. We consider the planar N -body problem with an attractive $1/r^\alpha$ potential for general masses. Let $q(t)$ be a relative equilibria, we show that the sectional curvature is zero along $q(t)$, for a certain set of planes containing $\dot{q}(t)$, if and only if $\alpha = 2$. Moreover, when $\alpha = 2$ we prove that a solution $q(t)$ with constant moment of inertia, has constant potential energy if and only if the sectional curvature is zero along the $q(t)$ for this certain set planes. Finally for any zero energy case, we show that the holomorphic sectional curvature is non-positive and vanishes exactly along the relative equilibria.

1. INTRODUCTION

Upon fixing the energy, the Jacobi-Maupertuis principle reparametrizes solutions of a natural Hamiltonian system as geodesics of a certain metric [see Eq. (2)] which we call the Jacobi-Maupertuis or JM-metric for short. In principle knowledge of the curvature of this metric should then lead to some dynamical consequences, as R. Montgomery obtained for the case of three-bodies in [6]. More recently, in [2] the authors have also found some interesting results about the curvature of this geodesic flow in the reduced space on the collinear and parallelogram surfaces for four-bodies. Here we see how the curvature of this JM-metric on the configuration space relates to relative equilibria. In particular, we also consider the sectional curvature in the realm of complex planes to give a characterisation of them.

2. NOTATIONS AND OUTLINE OF RESULTS

Consider N point particles of positive masses m_k and positions $q_k \in \mathbb{R}^2$. We will identify the Euclidean plane \mathbb{R}^2 with the complex numbers \mathbb{C} . The configuration of the system is described by the vector

$$q = (q_1, \dots, q_N) \in \mathbb{C}^N \setminus \Delta$$

where

$$\Delta = \{q = (q_1, \dots, q_N) \in \mathbb{C}^N : q_k = q_l, k \neq l\}$$

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consists of all the collisions.

The mass metric on configuration space is the Hermitian inner product

$$\langle u, v \rangle_{\mathbb{C}} = \sum_{k=1}^N m_k \bar{u}_k v_k.$$

It is known that the complex vector space \mathbb{C}^N becomes a real vector space when we allow only scalar multiplication by real scalars. Besides, the real part and imaginary part of the Hermitian product are respectively the mass inner product and the mass symplectic structure:

$$\langle u, v \rangle = \operatorname{Re} \langle u, v \rangle_{\mathbb{C}} \quad \omega(u, v) = \operatorname{Im} \langle u, v \rangle_{\mathbb{C}}.$$

Thus, the equations of motion that we will consider are given by

$$(1) \quad \ddot{q}_k = \nabla_{q_k} U(q), \quad k = 1, \dots, N$$

where ∇ is the gradient for the mass metric and $U(q)$ is the (negative) potential function given by

$$U = \sum_{i < j} \frac{m_i m_j}{r_{ij}^\alpha}, \quad \alpha > 0$$

where $r_{ij} = |q_i - q_j|$ is the Euclidean distance between the masses m_i and m_j .

In general, one may assume that the center of mass is fixed at the origin of the system

$$\langle q, \mathbf{1} \rangle = m_1 q_1 + \dots + m_N q_N = 0.$$

Using the mass metric we have that the kinetic energy and the moment of inertia are expressed by

$$K(\dot{q}) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle = \frac{1}{2} \|\dot{q}\|^2$$

and

$$I(q) = \langle q, q \rangle = \|q\|^2$$

respectively. It is well known that the total energy

$$H(q, \dot{q}) = K(\dot{q}) - U(q)$$

is constant along solutions.

Also under the consideration about the center of mass, we can consider the relative equilibria as solutions of eq. (1) rotating around the origin with angular velocity $\omega \neq 0$, namely,

$$q_k(t) = e^{i\omega t} q_k(0), \quad k = 1, \dots, N.$$

We recall the Jacobi-Maupertuis reformulation of mechanics which asserts that the solutions to Newton's equations at energy H are, after a time reparameterization, precisely the geodesic equations for the Jacobi-Maupertuis metric

$$(2) \quad ds_{JM}^2 = 2(H + U)ds^2$$

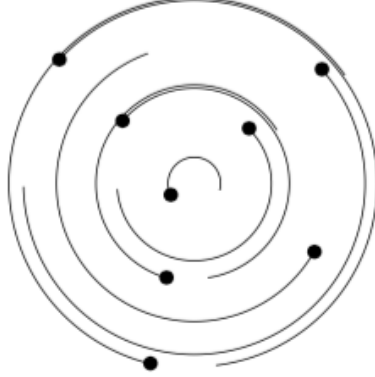


FIGURE 1. A relative equilibria solution. This figure was taken from [5]

on the Hill region $\{q \in \mathbb{C}^N \setminus \Delta : H + U \geq 0\} \subset \mathbb{C}^N \setminus \Delta$, where ds^2 is the mass metric.

From now we will consider the configuration space $\mathbb{C}^N \setminus \Delta$ endowed with the Jacobi-Maupertuis metric. Let $K_q(X, Y)$ denote the sectional curvature of the Jacobi-Maupertuis metric at q through the plane spanned by X and Y . Our results are the following.

Proposition 2.1. *Let $q(t)$ be a relative equilibria of the N -body problem with an attractive $1/r^\alpha$ potential. Then the following statements are equivalent*

- (1) *The sectional curvature $K_{q(t)}(\dot{q}(t), \mathbf{1})$ of the JM-metric is zero.*
- (2) *The negative potential U has degree -2 .*
- (3) *The total energy of the system is zero.*

Proposition 2.2. *Let $\alpha = 2$. Let $q(t)$ be a solution of the N -body problem with constant moment of inertia. Then its potential (or kinetic) energy is constant if and only if $K_{q(t)}(\dot{q}(t), \mathbf{1})$ is zero.*

Corollary 2.3. *Let $\alpha = 2$, $N = 3$. Let $q(t)$ be a solution of the N -body problem with constant moment of inertia. Then the sectional curvature of the Jacobi-Maupertuis metric along the $q(t)$ associated with the plane spanned by $\dot{q}(t)$ and $\mathbf{1}$ is zero if and only if $q(t)$ is a relative equilibria.*

Now we consider the curvature in the realm of complex planes and zero energy.

Proposition 2.4. *Let $q \in \mathbb{C}^N \setminus \Delta, v \in \mathbb{C}^N$ and consider the zero energy Jacobi-Maupertuis metric for an attractive $1/r^\alpha$ potential. Then the holomorphic sectional curvature at q along v (that is $K_q(v, iv)$) is non-positive.*

Moreover, for a solution $q(t)$ of eq. (1) we have the holomorphic sectional curvature over the $q(t)$ along $\dot{q}(t)$ is zero if and only if $q(t)$ is a relative equilibria.

QUESTION: When $\alpha = 2$, does the non-positive holomorphic sectional curvature persist on the reduced space, that is under the quotient by complex scaling?

3. PROOFS OF RESULTS

In order to establish our results, it will be necessary to use some auxiliary results. The first is on the sectional curvature of the configuration space.

Lemma 3.1 (Jackman and Montgomery in [3]). *The sectional curvature of $\mathbb{C}^N \setminus \Delta$ endowed with the Jacobi-Maupertuis metric $(H + U)ds^2$ is given by*

$$(3) \quad (H + U)^3 K(\sigma) = \frac{3}{4} ((\partial_1 U)^2 + (\partial_2 U)^2) - \left\| \frac{\nabla U}{2} \right\|^2 - \frac{H + U}{2} (\partial_1^2 U + \partial_2^2 U),$$

where $\partial_a U$ denotes $dU(v_a)$ and $a = 1, 2$ with v_1, v_2 are ds^2 -orthonormal vectors spanning $\sigma \subset \mathbb{C}^N$. The $\|\cdot\|$ and ∇ refer to the norm and Levi-Civita connection for the mass metric.

We are interested when the plane σ is spanned by the vectors $\dot{q}(t)$, $\mathbf{1}$.

Lemma 3.2. *Let $q(t)$ be a solution of the N -body problem with an attractive $1/r^\alpha$ potential. Then the sectional curvature of the Jacobi-Maupertuis metric along the $q(t)$ associated with the plane spanned by $\dot{q}(t)$ and $\mathbf{1}$ is given by*

$$(4) \quad 4\|\dot{q}\|^2 (H + U)^3 K_{q(t)}(\dot{q}(t), \mathbf{1}) = 3\langle \dot{q}, \ddot{q} \rangle^2 - \|\nabla U\|^2 \|\dot{q}\|^2 - 2(H + U) \langle \dot{q}, \ddot{q} \rangle$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ refer to the mass metric.

Proof. Let us consider the directions given by

$$v_1 = \frac{\dot{q}}{\|\dot{q}\|}, \quad v_2 = \frac{\mathbf{1}}{\|\mathbf{1}\|}.$$

It is direct to see that

$$(5) \quad \partial_1 U = \langle \nabla U, v_1 \rangle = \frac{\langle \nabla U, \dot{q} \rangle}{\|\dot{q}\|} = \frac{\langle \ddot{q}, \dot{q} \rangle}{\|\dot{q}\|}.$$

We consider the functions $g_k(t) := \nabla_{x_k} U(q(t)) = \ddot{x}_k(t)$, for $k = 1, \dots, 2N$, where $q = (x_1, \dots, x_{2N})$. Since

$$\nabla_{x_k} U = \frac{1}{m_k} \frac{\partial U}{\partial x_k}$$

we have

$$\dot{g}_k(t) = \frac{1}{m_k} \sum_{l=1}^{2N} \dot{x}_l(t) \frac{\partial^2 U}{\partial x_k \partial x_l}(q(t)) = \ddot{x}_k(t)$$

and

$$\langle \dot{q}, \ddot{q} \rangle = \langle \dot{q}, (\dot{g}_1, \dots, \dot{g}_{2N}) \rangle = \sum_{k,l=1}^{2N} \dot{x}_k(t) \dot{x}_l(t) \frac{\partial^2 U}{\partial x_k \partial x_l}(q(t)).$$

On the other hand, we also have that

$$\langle \nabla \langle \nabla U, \dot{q} \rangle, \dot{q} \rangle = \sum_{k,l=1}^{2N} \dot{x}_k(t) \dot{x}_l(t) \frac{\partial^2 U}{\partial x_k \partial x_l}(q(t)).$$

Thus, we obtain the relation $\langle \nabla \langle \nabla U, \dot{q} \rangle, \dot{q} \rangle = \langle \dot{q}, \ddot{q} \rangle$. Now we can determine the term $\partial_1^2 U$:

$$\begin{aligned} \partial_1^2 U &= \langle \nabla \langle \nabla U, v_1 \rangle, v_1 \rangle \\ &= \|\dot{q}\|^{-2} \langle \nabla \langle \nabla U, \dot{q} \rangle, \dot{q} \rangle. \end{aligned}$$

Thus,

$$(6) \quad \partial_1^2 U = \frac{\langle \dot{q}, \ddot{q} \rangle}{\|\dot{q}\|^2}.$$

Let us now consider the term $\partial_2 U$. Since $\nabla U \in \{q \in \mathbb{C}^N \setminus \Delta : \sum m_k q_k = 0\}$, we obtain

$$(7) \quad \partial_2 U = \langle \nabla U, v_2 \rangle = \frac{\langle \nabla U, \mathbf{1} \rangle}{\|\mathbf{1}\|} = 0.$$

Similarly, $\partial_2^2 U = 0$. Now substitution of (5), (6), (7) into (3) and multiplying by $4\|\dot{q}\|^2$ yields (4). \square

We will use the Lagrange-Jacobi identity for the homogeneous potential $-U$ of degree $-\alpha$:

$$(8) \quad \ddot{I} = 4H + (4 - 2\alpha)U$$

Proof of Proposition 2.1. Let $q(t) = e^{i\omega t} q_0$ be a relative equilibrium of the N -body problem with an attractive $1/r^\alpha$ potential. Note that $I(q(t)) = \text{constant}$. It follows from

$$\dot{q}(t) = i\omega q(t), \quad \ddot{q}(t) = -\omega^2 q(t), \quad \ddot{\bar{q}}(t) = -i\omega^3 \bar{q}(t)$$

that

$$\langle \dot{q}, \ddot{q} \rangle = 0, \quad \|\dot{q}\|^2 = \omega^2 \|q_0\|^2, \quad \langle \dot{q}, \ddot{\bar{q}} \rangle = -\omega^4 \|q_0\|^2$$

and

$$\|\nabla U(q)\|^2 = \|\ddot{q}\|^2 = \omega^4 \|q_0\|^2, \quad 2K = \omega^2 \|q_0\|^2.$$

Using Lemma 3.2, we get that $K_q(\dot{q}, \mathbf{1})$ is zero if and only if

$$-\omega^2 \|q_0\|^2 + 2(H + U) = 0$$

which is equivalent to that the total energy H is zero. Finally, by the Lagrange-Jacobi identity, with $\ddot{I} = 0$, it is immediate to see that $H = 0$ if and only if $\alpha = 2$. \square

Lemma 3.3. *Let $q(t)$ be a solution of the N -body problem with an attractive $1/r^\alpha$ potential with total energy zero. Then the sectional curvature of the Jacobi-Maupertuis metric along the $q(t)$ associated with the plane spanned by $\dot{q}(t)$ and $\mathbf{1}$ is given by*

$$(9) \quad 8U^4 K_{q(t)}(\dot{q}(t), \mathbf{1}) = 3 \left(\frac{dU}{dt} \right)^2 - 2U \frac{d^2U}{dt^2}.$$

Proof. Since total energy is zero, we obtain $U = K = \frac{1}{2} \|\dot{q}\|^2$,

$$\frac{dU}{dt} = \langle \dot{q}, \ddot{q} \rangle, \quad \text{and} \quad \frac{d^2U}{dt^2} = \langle \ddot{q}, \ddot{q} \rangle + \langle \dot{q}, \ddot{\ddot{q}} \rangle.$$

By Lemma 3.2, this implies that

$$4\|\dot{q}\|^2 U^3 K_q(\sigma) = 3 \left(\frac{dU}{dt} \right)^2 - \|\dot{q}\|^2 \left(\frac{d^2U}{dt^2} \right).$$

Again using that $\|\dot{q}\|^2 = 2U$, we obtain (9). \square

Proof of Proposition 2.2. Let $q(t)$ be a solution of N -body problem with constant moment of inertia. From the Lagrange-Jacobi identity, we obtain that the total energy is zero, and we can use Lemma 3.3. Thus, by (9), it is enough to see that if $K_{q(t)}(\dot{q}(t), \mathbf{1}) = 0$ we obtain that $U(q(t))$ is constant.

We suppose, for the sake of contradiction, that $\frac{dU}{dt} \neq 0$ for some t . It is straightforward to see that $K_{q(t)}(\dot{q}(t), \mathbf{1}) = 0$ is equivalent to its first integral

$$U^3 = C \left(\frac{dU}{dt} \right)^2,$$

where C is a constant. In particular, we have $\{U(q(t))\} = \mathbb{R}^+$. Then $U(q(t)) \rightarrow 0$ implies:

$$\limsup r_{ij} = \infty, \quad \text{for all } 1 \leq i < j \leq N,$$

giving a contradiction. Indeed, the moment of inertia can be written as $I = \frac{1}{M} \sum_{i < j} m_i m_j r_{ij}^2$,

$M = m_1 + \dots + m_N$, which by hypothesis is constant. It follows $U(q(t))$ must be constant. \square

Corollary 2.3 is an immediate consequence of [1, Theorem 1] and Proposition 2.2.

In order to establish Proposition 2.4, let us first set some background notations.

Given a Riemannian manifold, (M^{2n}, g) with metric compatible almost complex structure J , we split the complexified tangent space into the i , $-i$ eigenspaces of J 's extension, $J(v \otimes \lambda) := J(v) \otimes \lambda$:

$$TM \otimes \mathbb{C} = TM' \oplus TM''.$$

In some local coordinates (x^j, y^j) on M s.t. $J(\partial_{x^j}) = \partial_{y^j}$ we then have the bases for TM' and TM'' respectively as $\partial_j := \frac{1}{2}(\partial_{x^j} \otimes 1 - \partial_{y^j} \otimes i)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_{x^j} \otimes 1 + \partial_{y^j} \otimes i)$.

Now we extend the metric \mathbb{C} -linearly to a \mathbb{C} -valued symmetric bilinear form on $TM \otimes \mathbb{C}$, by $g(v \otimes \lambda, \cdot) := \lambda g(v, \cdot)$. Using the metric compatibility of J we find $g_{ij} = g(\partial_i, \partial_j) = 0 = g_{i\bar{j}}$ and

$$g_{i\bar{j}} = \frac{1}{2}(g(\partial_{x^i}, \partial_{x^j}) + ig(\partial_{x^i}, \partial_{y^j})) = g_{ij}$$

and so

$$g = g_{i\bar{j}}(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i) = 2g_{i\bar{j}}dz^i d\bar{z}^j$$

where $dz^i = dx^i + idy^i$, $d\bar{z}^i = dx^i - idy^i$ are dual to $\partial_i, \bar{\partial}_i$.

The process above with the usual identification of \mathbb{R}^2 with \mathbb{C} by $z = x + iy$ and $J(x, y) = (y, -x)$ corresponding to multiplication by i , yields the familiar operators:

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), & \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \\ dz &= dx + idy, & d\bar{z} &= dx - idy, \end{aligned}$$

With this notation note that

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}, \\ dz d\bar{z} &= dx^2 + dy^2. \end{aligned}$$

Let $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. It then follows that a function $z \mapsto f(z) \in \mathbb{C}$ is holomorphic if and only if $\bar{\partial}f(z) = 0$.

In the following we consider the JM-metric with $H = 0$.

Lemma 3.4. *Let $z \mapsto g(z) \in \mathbb{C}^N$ be holomorphic. Then*

$$(10) \quad -\partial \bar{\partial} \log \|g(z)\|^2 = \frac{|\langle g, g' \rangle|^2}{\|g\|^4} - \frac{\|g'\|^2}{\|g\|^2}$$

where $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ refer to the Hermitian mass metric while $|z|^2 = z\bar{z}$ is the usual \mathbb{C} -norm.

Note that by the Cauchy-Schwarz inequality, eq. (10) is non-positive and is zero exactly when $g' = \lambda g$ for some $\lambda \in \mathbb{C}$.

Proof. Write $g(z) = (g_1(z), \dots, g_N(z))$ for $g_i(z)$ holomorphic. As $\|g\|^2 = \sum m_k g_k \bar{g}_k$ we have

$$\bar{\partial} \log \|g\|^2 = \frac{\sum m_k g_k \bar{\partial} \bar{g}_k}{\|g\|^2} = \frac{\langle g, g' \rangle}{\|g\|^2}$$

and so

$$\begin{aligned}\partial\bar{\partial}\log\|g\|^2 &= \frac{\sum m_k \partial g_k \overline{\partial g_k}}{\|g\|^2} - \frac{\langle g, g' \rangle \sum m_k \partial g_k \bar{g}_k}{\|g\|^4} \\ &= \frac{\|g'\|^2}{\|g\|^2} - \frac{|\langle g, g' \rangle|^2}{\|g\|^4}.\end{aligned}$$

□

Lemma 3.5. *The Kobayashi Holomorphic sectional curvature, $H_q(X)$ (see [4], ch. 2 §3), of the $1/r^\alpha$ JM-metric is non-positive, moreover it vanishes exactly along $X \in \text{span}_{\mathbb{C}}\{q, \mathbf{1}\}$.*

Proof. The Kobayashi holomorphic sectional curvature is

$$H_q^{\text{hol}}(X) := \sup K_{f^* ds_{JM}^2}(0)$$

where the sup is taken over all holomorphic maps $f : D \rightarrow \mathbb{C}^N$ with $f(0) = q$ and $f'(0) \in \mathbb{C}X$. Here we write $K_{2\rho dz d\bar{z}} = -\frac{\partial\bar{\partial}\log\rho}{\rho}$ for the Gaussian curvature of the metric $2\rho dz d\bar{z}$ on the disk D .

Since

$$f^* ds_{JM}^2 = U(f(z)) \|f'(z)\|^2 dz d\bar{z} =: 2\rho dz d\bar{z},$$

we seek the supremum of:

$$\rho K_{f^* ds_{JM}^2} = -\partial\bar{\partial}\log U(f(z)) - \partial\bar{\partial}\log \|f'(z)\|^2$$

with everything to be evaluated at $z = 0$.

We can write the first term, $-\partial\bar{\partial}\log U(f(z))$, as

$$-\frac{\sum \partial_i \bar{\partial}_j U \partial f_i \bar{\partial} f_j}{U} + \frac{(\sum \partial_i U \partial f_i)(\sum \bar{\partial}_i U \bar{\partial} f_i)}{U^2}$$

for $\partial_i = \frac{\partial}{\partial q_i}$. Note that this contains only first derivatives of f since f is holomorphic. Moreover scaling $f'(0)$ to $\lambda f'(0)$ scales both $\partial\bar{\partial}\log U(f(z))$ and ρ by $|\lambda|^2$ and hence the first term is unaffected by the choice of f in the supremum.

Now by Lemma 3.4 the second term, $-\partial\bar{\partial}\log \|f'(z)\|^2$, has a supremum of zero which we may realize for instance when f is linear:

$$f(z) = q + zX.$$

Setting $g_{ij}(z) = \frac{\sqrt{m_i m_j}}{(q_i - q_j + z(X_i - X_j))^{\alpha/2}}$ we have

$$U(f(z)) = \sum_{i < j} |g_{ij}(z)|^2$$

and so again by Lemma 3.4, $-\partial\bar{\partial}\log U(f(z))$ is non-positive and is zero exactly when

$$\lambda g_{ij}(0) = g'_{ij}(0)$$

holds for some $\lambda \in \mathbb{C}$ and for all $i < j$. That is to say

$$(X_i - X_j) = \lambda_1(q_i - q_j)$$

for $\lambda_1 = -\frac{2}{\alpha}\lambda$ which implies that $X_i - \lambda_1 q_i$ is constant, say λ_2 . Hence we have

$$X = \lambda_1 q + \lambda_2 \mathbf{1}$$

as desired. \square

Lemma 3.6. *Consider our JM-metric $g = Uds^2$ with compatible almost complex structure J corresponding to multiplication by i . The Kobayashi holomorphic sectional curvature, $H_q(v)$ is related to the holomorphic sectional curvature, $K_q(v, Jv)$ by:*

$$(11) \quad K_q(v, Jv) = H_q(v) - \frac{\sum_{j=2}^N \partial_j U \bar{\partial}_j U}{U^3}$$

where ∂_j are some complex orthonormal basis for the mass metric ds^2 such that $\partial_1 = v/\|v\|$.

Proof. Let $v \in \mathbb{C}^N$ with Euclidean norm 1. Take an orthonormal Euclidean basis with $\tilde{v} = (1, 0, \dots, 0)$ and write the JM-metric as: $U \sum \mu_j dz_j d\bar{z}_j$ where $\mu_j > 0$ are some positive constants depending on the masses.

In the real coordinates (x_j, y_j) where $z_j = x_j + iy_j$ we then have:

$$K(v, Jv) = \frac{R_{x_1 y_1 x_1 y_1}}{\mu_1^2 U^2}$$

where R_{ijkl} is the Riemannian curvature tensor associated to the real metric $U \sum \mu_j (dx_j^2 + dy_j^2)$. We compute in these coordinates (or by eq. 4) that:

$$R_{x_1 y_1 x_1 y_1} = \frac{\mu_1}{2} \left(-\Delta_1 U + \frac{(\partial_{x_1} U)^2 + (\partial_{y_1} U)^2}{U} - \frac{\sum_{j=2}^N \frac{\mu_1}{\mu_j} ((\partial_{x_j} U)^2 + (\partial_{y_j} U)^2)}{2U} \right)$$

and takes the form in complex coordinates ($\partial_j = \frac{\partial}{\partial z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$)

$$R_{x_1 y_1 x_1 y_1} = \frac{\mu_1}{2} \left(-4\partial_1 \bar{\partial}_1 U + \frac{4\partial_1 U \bar{\partial}_1 U}{U} - \frac{2 \sum_{j=2}^N \frac{\mu_1}{\mu_j} \partial_j U \bar{\partial}_j U}{U} \right).$$

Kobayashi shows as well ([4] ch. 2) that his definition is equivalent to $H(X) = R_{i\bar{j}k\bar{l}} X^i \bar{X}^j X^k \bar{X}^l$ for a unit vector $X = X^j \partial_j$ (that is $g(X, \bar{X}) = 1$ by extending g \mathbb{C} -linearly) and where

$$R_{i\bar{j}k\bar{l}} = -\partial_k \bar{\partial}_l g_{i\bar{j}} + g^{p\bar{q}} \partial_k g_{i\bar{q}} \bar{\partial}_l g_{p\bar{j}}.$$

In our case we have $v = \partial_1$ and the corresponding unit vector is $X = \sqrt{\frac{2}{\mu_1 U}} \partial_1 = \frac{\partial_1}{\sqrt{g_{1\bar{1}}}}$. Now we compute:

$$R_{1\bar{1}1\bar{1}} = \frac{\mu_1}{2} (-\partial_1 \bar{\partial}_1 U + \frac{\partial_1 U \bar{\partial}_1 U}{U})$$

and then

$$H_q(v) = \frac{4R_{1\bar{1}1\bar{1}}}{\mu_1^2 U^2} = K_q(v, Jv) + \frac{\sum_{j=2}^N \frac{1}{\mu_j} \partial_j U \bar{\partial}_j U}{U^3}.$$

The formula in the Lemma then follows by rescaling ∂_j by $\frac{1}{\sqrt{\mu_j}}$.

□

Proof of Proposition 2.4. That the holomorphic sectional curvature is non-positive follows now since $H_q(v) \leq 0$ by Lemma 3.5 and $K_q(v, Jv) \leq H_q(v)$ by Lemma 3.6.

A *relative equilibrium solution* is one for which $\dot{q} \in \text{span}_{\mathbb{C}}\{q, \mathbf{1}\}$. Or equivalently $\nabla U = \lambda q$ for some $\lambda \in \mathbb{C}$.

First suppose $q(t)$ is a relative equilibrium solution. Then ∇U is contained in the \mathbb{C} -span of q i.e. the \mathbb{C} -span of ∂_1 according to the basis chosen in Lemma 3.6. Hence $\partial_j U = 0$ for $j \geq 2$ and

$$K_q(\dot{q}, J\dot{q}) = H_q(\lambda_1 q + \lambda_2 \mathbf{1}) = 0.$$

Now suppose $K_q(\dot{q}, J\dot{q}) = 0$ along some solution $q(t)$. Then $H_q(\dot{q}) \geq 0$ by Lemma 3.6, but now applying Lemma 3.5 we see that $H_q(\dot{q}) = 0$ and hence $\dot{q} \in \text{span}_{\mathbb{C}}\{q, \mathbf{1}\}$.

□

REMARKS: From the proof (in particular eq. (11)), we may rephrase Proposition 2.4 as $q(t)$ is a relative equilibrium if and only if the holomorphic sectional curvature and Kobayashi holomorphic sectional curvature agree over the solution which holds for any energy level $H = h$. Moreover a slight modification of the proof shows that the curvature over these relative equilibria is $-h \frac{|\lambda|^2 U}{(h+U)^3 |\dot{q}|^2}$, which for relative equilibria of the form $q = e^{i\omega t} q_0$ reads as $-h \frac{\alpha^2 U(q_0)}{4(h+U(q_0))^3}$.

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REFERENCES

- [1] Fujiwara, Toshiaki; Fukuda, Hiroshi; Ozaki, Hiroshi; Taniguchi, Tetsuya *Saari's homographic conjecture for general masses in planar three-body problem under Newton potential and a strong force potential*, J. Phys. A: Math. Theor. **48** (2015) 265–205.
- [2] C. Jackman and J. Meléndez. *Hyperbolic Shirts fit a 4-body problem*, Preprint 2016.

- [3] C. Jackman and R. Montgomery. *No hyperbolic pants for the 4-body problem with strong potential*, Pacific J. Math. **280** (2016), 401–410.
- [4] S. Kobayashi. *Hyperbolic Complex Spaces*, Springer Series of comprehensive studies in mathematics vol 318 (1998).
- [5] R. Moeckel. *Lectures on Central Configurations*.
- [6] R. Montgomery. *Fitting hyperbolic pants to a three-body problem*, Ergodic Theory Dynam. Systems **25** (2005), 921–947.

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